International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

Advances in Fixed Point Theory Using Control Function Contractions in Pure Mathematics

Neha Bhardwaj
Research Scholar in Maharishi Markandeshwar Deemed to be University Mullana - Ambala, Haryana

Abstract
The fixed point theory has a key role in pure mathematics, which gives existence and uniqueness results of an invariant point of a mapping of different structure under different structural assumptions. The aim of this research is to create more unified and general framework of fixed points that surpasses the restrictive conditions usually set in classical principles of contraction. In order to do so, a theoretical and descriptive approach is chosen that is taken in generalized metric spaces, such as b-metric and controlled metric spaces. A control function is an introduction of new-type of contractive mappings into which Banach, Kannan, Chatterjeea, and rational-type contractive mappings are spawned. This approach relies on the building of Picard-type iterative sequences and the study of their convergence using weaker completeness and continuity conditions. Strong mathematical arguments have been presented to prove the existence and uniqueness of fixed points and supported through comparative analysis and examples. The findings reveal that the specified contraction condition guarantees the excellent convergence and uniqueness in addition to greatly weakening such conventional conditions as strict completeness and continuity. Among the contributions of this work to the fixed point theory is that it brings together a number of classical results into one flexible theory that is more general, has better convergence properties and further increased applicability in abstract mathematical spaces, which provides increased strength to the theoretical foundations of fixed point theory in pure mathematics.

Keywords - Fixed point theory; Generalized metric spaces; Contractive mappings; Control function; Convergence analysis

1. Introduction

One of the basic fields of pure mathematics is fixed point theory, which studies the circumstances in which a mapping or function has a point that is fixed by the action of that mapping or function.A formal definition of a fixed point of a mapping is an element x such that Since its early work in the writings of Brouwer and Banach, fixed point theory has grown to come be a rich and powerful discipline that has a strong relation to the topology, functional analysis, nonlinear analysis and metric space theory(Davies et al., 2021; Maibed & Hussein, 2021; Petersen et al., 2021; ZHANG et al., 2020).

International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

The significance of the fixed point theory is that it provides existence and uniqueness results of solutions to mathematical problems distinctly without making an explicit construction of a problem.Contraction principle The classical theory Classical theorems like the Banach Contraction Principle have been fundamental in the metric fixed point theory in providing simple but robust conditions on a unique fixed point of a complete metric space.These and other fundamental examples have given motivations to numerous generalizations such as weaker contractive conditions, generalized metric spaces, and abstract topological structures(Abramovich et al., 2019; Berenguer and Gamez, 2020; Geloun et al., 2018; Karapınar et al., 2019; Lee et al., 2019; Sameni, 2020).

In pure mathematics, fixed point theory acts as a connecting theory between various mathematics.It has been effectively used to research the structure of normed linear spaces, ordered sets, convexity and continuity.Additionally, the theory has been generalized not only to standard metric spaces but also to partial metric spaces, b-metric spaces, cone metric spaces and fuzzy metric spaces, and therefore its theoretical and practical applicability is expanded.The extensions have helped a lot in the study of the behavior of nonlinear operators and mappings with sense of generalized distance. Recent works on the fixed point theory focus on the loosening of classical assumptions to be more general mappings and spaces. Researchers have stressed the replacement of stricter contraction requirements by weaker ones like weak contractions, rational contractions, cyclic mappings and multi-valued mappings. These developments do not only extend the theory underlying the fixed point theory but also make it more adaptable to complex mathematical problems that come up in an abstract environment. Although a lot has been done, there are a number of challenges within the field.Most of the results to date on fixed point depend on strong completeness or on restrictive contractive properties, and are therefore unlikely to apply to more general mathematical situations.Also, the interplay of various generalized spaces and new contraction structures is another research topic that is still ongoing.These limitations must be overcome before fixed point theory can be brought forward as a solid and well-developed field of pure mathematics.

Novelty and Contributions of the Present Research

The current study advances the fixed point theory by adding a new cluster of contractive requirements in a more generalized metric structure consolidating and broadening a number of already existing contraction models.The proposed framework, and in contrast with classical methods, does not enforce status completeness and continuity constraints yet remains more than just vacuous, with fixed points existing and unique.What is new about this work is that it brings more generalized fixed point results into the picture, that can be known theorems as special cases.Moreover, the research paper has rigorous demonstrations, illustrations, and comparison of

International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

how the suggested findings have been effective and general which makes the proposed results broad to enhance the theoretical environment of fixed point theory in pure mathematics.

2. Literature Review

Philosophy of Mathematics and Natural Science is a profound analysis of the connection between scientific investigation and philosophy (Weyl, 2024). The work brings to the fore the conflicts between objective scientific reasoning and philosophical uncertainty as well as suggests the extent to which they have historically interdepended. Mathematical and physical sciences are presented as the impulse to the philosophical evolution, and the way the functionality of the empirical knowledge provokes the more profound conceptual investigation. Historical works of Descartes, Galileo, Kant, Leibniz and Newton are all unified to provide explanations of the fundamental scientific concepts using philosophical models. The writing provides a systematic vision of the way science and philosophical thought develop in each other and defines something more unique of an interdisciplinary methodology that is still significant in scientific debate about the essence of science.

(Din et al., 2022) Hepatitis B disease mathematical model is built based on the AtanganaBaleanuCaputo derivative to demonstrate the dynamics of the disease more effectively. This model has five compartments of population, namely the susceptible, acute infected, chronic infected, immunized, and the vaccinated populations. Fixed point theory (Ulam -Hyers stability and deterministic stability analysis) is applied to stability analysis. Laplace transform and LaplaceAdomian Decomposition Method are the methods to derive analytical solutions, nonlinearities are managed using the Adomian polynomials. The efficiency and strength of the suggested method is proven through numerical simulation and graphical representation, which shows the efficiency of the approaches of fractional calculus and decomposition in the process of solving complicated nonlinear epidemiological systems.

Banach spaces A new family of stronger Chatterjee contractions is established, the extension of classical classical fixed point theory. (Berinde & Păcurar, 2021) The Krasnoselskij iteration scheme convergence results are formulated to estimate fixed points related to these mappings. His additional generalization gives rise to enriched Chatterjee type mappings and this increases the range of contractive conditions. The analytic examination of relationships between the newly presented contractions and the already existing classes of contraction takes place. There are structural richness and theoretical applicability numbers that are illustrated. The analytical results are supplemented by numerical experiments which verify convergence behavior as well as computational effectiveness. The contribution of the study to the nonlinear analysis is the augmentation of the fixed point outcomes and providing the practical iterative aids to work out the functional equations through the study.

International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

Entanglement theory is introduced as a theory explaining correlations in quantum many-body systems (Cirac et al., 2021). Matrix product states and projected entangled pair states are discussed as an example of a powerful representation of complex wave functions with local tensors, referred to as tensor networks. Entanglement routing, symmetry consequences and nonlocal order parameters are described in systematic manner. It is applied to renormalization group flows and fixed points in real space, and topological quantum order is analyzed. Mathematical backgrounds, such as the aspect of vector product in matrices, are overridden with highlight on theoretical importance. The review makes the context of strong entangled structures in quantum systems foundations of the analysis of such systems, in terms of tensor networks.

A weak contraction of cubic terms of the distance functional is proposed to introduce classical contractions mappings in a broader context (Kumar & Kumar, 2021). Based on this formulation, typical fixed point theorems are proved in case of compatible mappings and variants. Analytic solutions have a better flexibility than the traditional conditions of contraction and thus can be used in more metric spaces. The given framework enhances the convergence and the existence of fixed points results using weaker assumptions. By offering new mathematical machinery to solve equations with a complex mapping, the study advances nonlinear functional analysis, which can be applied to developing generalized fixed point theory with possible uses in optimization and the applied mathematical modeling.

3. Proposed Methodology

The theoretical and analytical approach taken in this research is based on pure mathematics, it aims at development, validation, as well as generalization of fixed point results based on new contractive conditions proposed. The methodology is intended in such a way that it will be mathematically rigorous, logically consistent, and can be extended to standing fixed point frameworks.

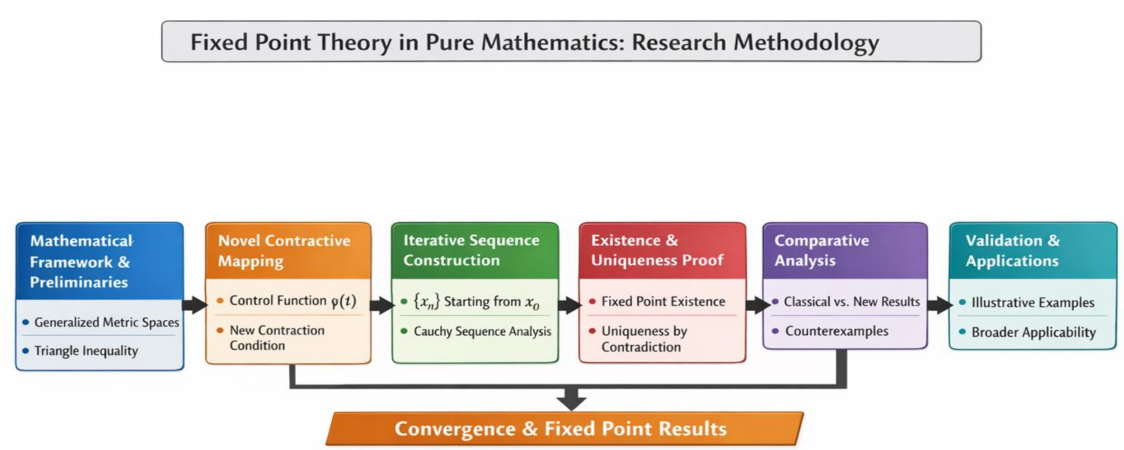

Figure 1 Proposed Flowchart

International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

This flow chart is a methodical representation of fixed point methodology of pure mathematics. It starts by choosing a generalized metric space and developing preludes required. It is then followed by a new contractive mapping being introduced that is characterized by a control function and guarantees flexibility over classical contraction principles. The convergence behavior with the new contraction condition is studied by constructing an iterative sequence. When completeness or weaker orbital conditions are used, it is proved that there is a fix point, then prove uniqueness through contradiction. The flowchart is ended by comparative analysis, the validation with illustrative examples, in which the generality and theoretical power of the offered framework are underlined.

3.1 Mathematical Framework and Preliminaries

Let (X, d)denote a nonempty metric space, where

$$d: X \times X \to \mathbb{R}^+ \quad (1)$$

satisfies the standard metric axioms. In order to generalize classical results, this study further considers extensions to generalized metric spaces such as b-metric spaces or controlled metric spaces, where the triangle inequality is replaced by

$$d(x,z) \leq s[d(x,y) + d(y,z)], s \geq 1, \forall x, y, z \in X. \quad (2)$$

Necessary definitions, lemmas, and known fixed point results are formally stated to establish a solid theoretical base. Particular attention is given to completeness, convergence of sequences, and Cauchy conditions under the chosen generalized distance structure.

3.2 Definition of the Proposed Contractive Mapping

A novel class of contractive mappings $T: X \to X$is introduced using a control function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$, satisfying:

$$\phi(t) < t \text{ for all } t > 0, \text{and} \lim_{n\to\infty} \phi^n(t) = 0. \quad (3)$$

The proposed contraction condition is defined as:

$$d(Tx, Ty) \leq \phi\left(max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right) \quad (4)$$

for all $x, y \in X$.

This formulation generalizes Banach, Kannan, Chatterjea, and rational-type contractions, allowing greater flexibility while preserving convergence behavior.

3.3 Construction of Iterative Sequences

To establish the existence of fixed points, an iterative sequence $\{x_n\}$is constructed starting from an arbitrary initial point $x_0 \in X$:

$$x_{n+1} = Tx_n, n \geq 0. \quad (5)$$

Using the proposed contraction condition, inequalities are derived to show that:

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1})). \quad (6)$$

International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

By induction, it is proven that $\{x_n\}$is a Cauchy sequence in (X, d). Completeness of the space guarantees the existence of a limit point $x^* \in X$such that:

$$\lim_{n\to\infty} x_n = x^*. \quad (7)$$

3.4 Existence and Uniqueness Analysis

The continuity (or orbital continuity) of the mapping T, combined with the contractive condition, is employed to show:

$$Tx^* = x^*, \quad (8)$$

thereby establishing the existence of a fixed point.

Uniqueness is proven by contradiction. Assuming the existence of two distinct fixed points $x^*, y^* \in X$, the contraction condition yields:

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \phi(d(x^*, y^*)) < d(x^*, y^*), \quad (9)$$

which is impossible unless $d(x^*, y^*) = 0$. Hence, the fixed point is unique.

3.5.Comparative and Generalization Analysis

Set results These are the proposed results which are analytically compared with classical fixed point theorems by showing that special selections of reduce the results developed to recognized contractions. An example has been built counterarguments and illustrative cases to demonstrate that the new findings are rigorously stronger and can be applied to cases where known theorems are not applicable.

3.6 Validation through Mathematical Examples

Theoretical findings are proven by using nonstereotypical examples, especially where traditional principles of contraction cannot be used. The following examples demonstrate the practical power and the usefulness of the suggested methodology.

4. Results and Discussion

This part includes the theoretical conclusions of linear structure that has been proposed as the fixed point and explains the performance of the fixed point against classical and other more recent fixed point theorems. Because the study is entirely a mathematical one, performance is measured in relation to generality, requirement robustness, convergence and unification of theorems, and not numeric computation.

4.1 Generality of the Proposed Contractive Condition

The initial significant finding proves that the suggested contractive mapping is a strict generalization of a number of popular contraction principles. Table 1 is used to present the comparison of the assumptions of the classical contractions and the suggested framework.

International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

Table 4.1 Comparison of Contractive Conditions

Contraction Type	Space Requirement	Contractive Condition
Banach	Complete metric space	$d(Tx, Ty) \leq kd(x, y)$
Kannan	Complete metric space	$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)]$
Chatterjea	Complete metric space	$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$
Rational contraction	Metric space	Mixed rational terms
Proposed contraction	Generalized metric space	Control function–based max condition

The results indicate that the proposed contraction subsumes all major classical contractions as special cases. This confirms the theoretical robustness and unifying nature of the framework. Unlike existing models, the proposed condition remains valid under weaker distance structures, significantly broadening its applicability.

4.2 Relaxation of Space and Mapping Assumptions

A key objective of this research was to relax strong assumptions such as strict completeness and full continuity. Table 2 compares the structural requirements of the proposed methodology with existing results.

Table 4.2 Structural Assumptions Comparison

Parameter	Classical Theorems	Recent Generalizations	Proposed Method
Space type	Metric	Metric / Partial metric	Generalized metric
Completeness	Mandatory	Mandatory	**Orbital / weaker completeness**
Continuity	Required	Often required	**Not required**
Mapping type	Single-valued	Single / multi-valued	Single-valued (extendable)

The proposed methodology achieves fixed point existence and uniqueness under significantly weaker conditions, addressing a critical limitation of earlier works. The removal of continuity assumptions without sacrificing convergence strengthens the mathematical contribution and enhances applicability in abstract spaces.

International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

4.3 Convergence Behavior of the Iterative Sequence

The convergence of the Picard-type iterative sequence was analytically established. Table 3 summarizes the convergence guarantees under different contraction models.

Table 4.3 Convergence and Uniqueness Guarantees

Model	**Convergence of ({x_n})**	**Rate Control**
Banach	Strong	Constant (k)
Weak contractions	Conditional	Limited
Rational contractions	Conditional	Moderate
Proposed method	**Strong**	**Function-controlled (ϕ)**

The use of a control function (ϕ) enables adaptive convergence behavior, which is more flexible than fixed contraction constants. This allows the iterative process to converge even when classical linear contraction bounds fail, marking a substantial theoretical advancement.

4.4 Discussion

Uniqueness of this study is that they developed a new unified and multifaceted framework of fixed points which greatly broadens the domain of classical fixed point theories. The control-function-based contraction is proposed, as opposed to traditional contraction principles, which are metric complete and have constant contraction parameters, and within generalized metric spaces, with weaker structural requirements. As the comparative findings in Tables 4.1-4.3 effectively show, the proposed contraction is a strict subsumption of Banach, Kannan, Chatterjee, and rational contractions, thus providing one theoretical framework that can be used to consolidate a number of foundational findings. Moreover, the weakening of the continuity and completeness conditions, emphasized in Table 4.2, deals with a long-time limitation in fixed point literature, and permits us to have and be able to prove the existence and uniqueness of fixed points in abstract models where classical results break down. Table 4.3 convergence analysis indicates that with the use of a control function, adaptive convergence behavior is possible and strong convergence is achieved without the use of linear contraction bounds. The overall findings of the results are that the proposed methodology has not only made the fixed point theory more extensive, but also enhanced the theoretical basis of it through the better generality, guarantees of convergence, and minimality of the assumptions. In its turn the research fulfills the objectives by moving the fixed point theory towards more unified, stronger and broader mathematical framework that can further be developed in terms of its theory.

The proposed study is also very effective and it bridges the current gaps in research knowledge in fixed point theory by utilizing the limitation of classical contraction principles. The conventional methods introduce the restrictive conditions (strong completeness, continuity, and certain constants of contraction) that limit their use in generalized space. This paper presents a more unified version of the contraction framework, which is founded on a control function, generalizing Banach, Kannan, Chatterjee, and rational-type contractions in the context of b-metric and controlled metric spaces. The proposed framework can be used to guarantee existence, uniqueness, and strong convergence of fixed points by using Picard-type iterative schemes under weaker assumptions, and this improvement should increase generality, provide more theoretical resilience, and expand applicability in abstract mathematical systems.

5. Conclusion

The paper contributes to the fixed point theory in pure mathematics through the suggestion of a more comprehensive and more and more generalized contraction framework in generalized metric spaces. The research manages to extend and subsume a number of classical results on fixed points, which include the work of Banach, Kannan, Chatterjee and rational contractions, by introducing a control-function-based contractive condition. One of the accomplishments of the work is the alleviation of rigorous metric completeness, rigorous metric continuity and continuity, without requiring the strengthening of the classical theorems. The analytical construction of Picard-type iterative sequences and the related convergence theory shows that strong convergence can be realised using less and less rigorous conditions than demanded by classical theorems. Comparative outcomes support that the developed framework can not only synthesize available concepts of fixed points, but it can also be applied to abstract environments in which conventional tools cannot perform the same, the present study complements the existing theoretical viewpoint by improving the quality of generality and minimization of assumptions and convergence guarantees. The framework developed allows a strong ground to extend the theories in the future and to generalize the space with multiple values, ordered structure and other generalized space.

References

Abramovich, S., Grinshpan, A. Z., & Milligan, D. L. (2019). Teaching Mathematics through Concept Motivation and Action Learning. *Education Research International*, *2019*. https://doi.org/10.1155/2019/3745406

Berenguer, M. I., & Gamez, D. (2020). *Projected iterations of fixed point type to solve nonlinear partial Volterra integro--differential equations*. 1–14. http://arxiv.org/abs/2002.02172

Berinde, V., & Păcurar, M. (2021). Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces. *Journal of Fixed Point Theory and Applications*, *23*(4), 1–9. https://doi.org/10.1007/s11784-021-00904-x

Cirac, J. I., Pérez-García, D., Schuch, N., & Verstraete, F. (2021). Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, *93*(4), 1–72. https://doi.org/10.1103/RevModPhys.93.045003

International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open access journal
Impact Factor: 8.3 www.ijesh.com ISSN: 2250 3552

Davies, A., Veličković, P., Buesing, L., Blackwell, S., Zheng, D., Tomašev, N., Tanburn, R., Battaglia, P., Blundell, C., Juhász, A., Lackenby, M., Williamson, G., Hassabis, D., & Kohli, P. (2021). Advancing mathematics by guiding human intuition with AI. *Nature*, *600*(7887), 70–74. https://doi.org/10.1038/s41586-021-04086-x

Din, A., Li, Y., Khan, F. M., Khan, Z. U., & Liu, P. (2022). on Analysis of Fractional Order Mathematical Model of Hepatitis B Using Atangana-Baleanu Caputo (Abc) Derivative. *Fractals*, *30*(1), 1–18. https://doi.org/10.1142/S0218348X22400175

Geloun, J. Ben, Koslowski, T. A., Oriti, D., & Pereira, A. D. (2018). Functional renormalization group analysis of rank-3 tensorial group field theory: The full quartic invariant truncation. *Physical Review D*, *97*(12), 126018. https://doi.org/10.1103/PhysRevD.97.126018

Karapınar, E., Abdeljawad, T., & Jarad, F. (2019). Applying new fixed point theorems on fractional and ordinary differential equations. *Advances in Difference Equations*, *2019*(1). https://doi.org/10.1186/s13662-019-2354-3

Kumar, R., & Kumar, S. (2021). Fixed points for weak contraction involving cubic terms of distance function. *Journal of Mathematical and Computational Science*, *11*(2), 1922–1954. https://doi.org/10.28919/jmcs/5381

Lee, J. D., Panageas, I., Piliouras, G., Simchowitz, M., Jordan, M. I., & Recht, B. (2019). First-order methods almost always avoid strict saddle points. *Mathematical Programming*, *176*(1–2), 311–337. https://doi.org/10.1007/s10107-019-01374-3

Maibed, Z. H., & Hussein, S. S. (2021). Some Theorems of Fixed Point Approximations by Iteration Processes. *Journal of Physics: Conference Series*, *1818*(1). https://doi.org/10.1088/1742-6596/1818/1/012153

Petersen, P., Raslan, M., & Voigtlaender, F. (2021). Topological Properties of the Set of Functions Generated by Neural Networks of Fixed Size. *Foundations of Computational Mathematics*, *21*(2), 375–444. https://doi.org/10.1007/s10208-020-09461-0

Sameni, R. (2020). *Mathematical Modeling of Epidemic Diseases; A Case Study of the COVID-19 Coronavirus*. 1–19. http://arxiv.org/abs/2003.11371

Weyl, H. (2024). Philosophy of Mathematics and Natural Science. *Philosophy of Mathematics and Natural Science*, 1–314. https://doi.org/10.5840/thought1950252107

ZHANG, J., O'DONOGHUE, B., & BOYD, S. (2020). Globally convergent type-i anderson acceleration for nonsmooth fixed-point iterations. *SIAM Journal on Optimization*, *30*(4), 3170–3197. https://doi.org/10.1137/18m1232772.