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Attain A Unique Fixed Point by Drawing Contractive Type Condition and Control Function in Fuzzy Metric Space

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Abstract

In this paper, we attain a unique fixed point for contractive mappings in fuzzy metric spaces by employing a suitable control function and a contractive-type inequality. The obtained results generalize several existing fixed point theorems and ensure existence, uniqueness, and convergence of fixed points under weaker conditions.

Keywords and Phrases: Control Function, Contractive Condition, Complete Fuzzy Metric Space , Fixed Point Theorem, Integral Type, Rational Expression., AMS (2010) Subject Classifications: Primary 54H25, Secondary 47H10.

1. Introduction And Preliminaries

Known by every mathematician that the concept of fuzzy metric space (FMS)introduced , initiated and devolved by zadeh (1965) after all namely E l Naschie (2004), Gupta et.al (2015), Grabeic (1988) and also some other result of literature such as Vasuki (1998) , Gregori and Sapena (2002), Gupta and Mani (2014a) and Gupta,et.al.(2015) , Banach (1922), George & Veeramani, (1994), Grabiec(1988), Gregori, Morillas & Sapena (2011), Vijayaraju & Sajath, (2009), Subrahmanyam, (1995), Karmosill, & Michalek (1975), Saini, Gupta, & Singh, (2007), Saini.et.al (2008), Schweizer & Sklar (1960).

Again some other researcher such as Jain and Sayyed (2019), Gupta, Saini, Mani & Tripathi, (2015) , Gupta & Mani (2014b) also give some theory on control fuzzy metric space.

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2. Main Results

THEOREM 2.1 Let (X, M, *) be a complete fuzzy metric space and p : X → X be a mapping satisfying

$$M(px, py, kt) \geq \xi \{ \lambda (x, y, t) \} \qquad \text{--- (A)}$$

Where,

$$\lambda (x, y, t) = \left\{ M(x, y, t) + M(x, px, t) + M(y, py, t) + \frac{M(x, px, t) * M(y, py, t)}{M(x, y, t)} \right\} \qquad \text{--- (A')}$$

for all x, y ∈ X, $\xi \in \phi$ and k ∈ (0, 1). Then p has a unique fixed point.

PROOF: Let Construct a sequence $\{X_n\} \in X$ such that $px_n = x_{n+1}$ for all n ∈ N, where x ∈ X be any arbitrary point in X. Claim: $\{X_n\}$ is a Cauchy sequence with $x = x_{n-1}$ and $y = x_n$ in equation A, we get

$$M(x_n, x_{n+1}, kt) = M(px_{n-1}, px_n, kt) \geq, \xi \{ \lambda (x_{n-1}, x_n, t)$$

from equation A', we have

$$\lambda (x_{n-1}, x_n, t)$$

$$= \left\{ M(x_{n-1}, x_n, t) + M(x_{n-1}, px_{n-1}, t) + M(x_n, px_n, t) + \frac{M(x_{n-1}, px_{n-1}, t) + M(x_n, px_n, t)}{M(x_{n-1}, x_n, t)} \right\}$$

$$= \left\{ M(x_{n-1}, x_n, t) + M(x_{n-1}, x_n, t) + M(x_n, x_{n+1}, t) + \frac{M(x_{n-1}, x_n, t) + M(x_n, x_{n+1}, t)}{M(x_{n-1}, x_n, t)} \right\}$$

$$= 2\{ M(x_n, x_{n+1}, t) + M(x_{n-1}, x_n, t) \}$$

Now if $M(x_n, x_{n+1}, t) \leq M(x_{n-1}, x_n, t)$ then by above,

$$M(x_n, x_{n+1}, kt) \geq \xi \{M(x_n, x_{n+1}, t)\} > M(x_n, x_{n+1}, t)$$

Hence, our claim follows immediately from previous obtained lemma , Now suppose

$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$, then again from ,

$$M(x_n, x_{n+1}, kt) \geq \xi \{M(x_{n-1}, x_n, t)\} > M(x_{n-1}, x_n, t)$$

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Now by simple induction, for all n and t > 0, we get

$M(x_n, x_{n+1}, kt) \geq M(x, x_1, \frac{t}{k^{n-1}})$

Now for any positive integer r, we have

$M(x_n, x_{n+r}, t) \geq M(x_n, x_{n+1}, \frac{t}{r}) * \ldots\ldots\ldots * M(x_{n+p-1}, x_{n+p}, \frac{t}{r})$, proceeding

$M(x_n, x_{n+r}, t) \geq M(x, x_1, \frac{t}{r\,k^n}) * \text{---} . * M(x, x, \frac{t}{r\,k^n})$

Taking $\lim_{n\to\infty}$, we get $\lim_{n\to\infty} M(x_n, x_{n+r}, t) = 1$

This implies $\{x_n\}$ is a Cauchy Sequence, therefore, there exists a point u ∈ X such that

$\lim_{n\to\infty} x_n = u.$

Claim: u is a fixed point of p. Consider

$M(u, pu, t) \geq M(px_n, pu, t) * M(u, x_{n+1}, t) \geq \xi\,\{\lambda\,(x_n, u, \frac{t}{2k})\} * M(u, x_{n+1}, t)$ ---(2.6)

Again from Equation A',

$\lambda\,(x_n, u, \frac{t}{2k}) =$

$$\left\{M\left(x_n, u, \frac{t}{2k}\right) + M\left(x_n, px_n, \frac{t}{2k}\right) + M\left(u, pu, \frac{t}{2k}\right) + \frac{M\left(x_n, px_n, \frac{t}{2k}\right) * M\left(u, pu, \frac{t}{2k}\right)}{\left(x_n, u, \frac{t}{2k}\right)}\right\}$$

Taking $\lim_{n\to\infty}$, we get

$\lambda\,(u, u, \frac{t}{2k}) =$

$$\left\{M\left(u, u, \frac{t}{2k}\right) + M\left(u, pu, \frac{t}{2k}\right) + M\left(u, pu, \frac{t}{2k}\right) + \frac{M\left(u, pu, \frac{t}{2k}\right) * M\left(u, pu, \frac{t}{2k}\right)}{\left(u, u, \frac{t}{2k}\right)}\right\}$$

$$= \left\{1 + M\left(u, pu, \frac{t}{2k}\right) + M\left(u, pu, \frac{t}{2k}\right) + \left(M\left(u, pu, \frac{t}{2k}\right)\right)^2\right\}$$

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$= 1 + 2\,M\left(u, pu, \frac{t}{2k}\right) + \left(\mathrm{M}\left(\mathrm{u}, pu, \frac{t}{2k}\right)\right)^2$

By above equation , we get

$\mathrm{M}\ (\mathrm{u}, \mathrm{pu}, \mathrm{t}) \geq \xi\left\{M\left(u, pu, \frac{t}{2k}\right)\right\} * \mathrm{M}\ (\mathrm{x}_{\mathrm{n}+1}, u, t) > \mathrm{M}\left(u, pu, \frac{t}{2k}\right) * \mathrm{M}\ (\mathrm{x}_{\mathrm{n}+1}, u, t)$

Taking $\lim_{n\to\infty}$ and using lemma of previous published papers reference we get pu = u.

UNIQUENESS: Now we show that u is a unique fixed point of p. Suppose not, then there exists a point z ∈ X such that pz = z.

Consider $1 \geq \mathrm{M}\ (\mathrm{z}, \mathrm{u}, \mathrm{t}) = \mathrm{M}\ (\mathrm{pz}, \mathrm{pu}, \mathrm{t}) \geq \xi\left\{\lambda\left(\mathrm{z}, \mathrm{u}, \frac{t}{k}\right)\right\}$

Where $\lambda\ (z, u, \frac{t}{k}) =$

$$\left\{M\left(z, u, \frac{t}{k}\right) + M\left(z, pz, \frac{t}{k}\right) + M\left(u, pu, \frac{t}{k}\right) + \frac{\mathrm{M}\left(z, pz, \frac{t}{k}\right) * \mathrm{M}\left(\mathrm{u}, \ \mathrm{pu}, \frac{t}{k}\right)}{M\left(z, \ u, \ \frac{t}{k}\right)}\right\}$$

$$= \left\{M\left(z, u, \frac{t}{k}\right) + M\left(z, z, \frac{t}{k}\right) + M\left(u, u, \frac{t}{k}\right) + \frac{\mathrm{M}\left(z, z, \frac{t}{k}\right) * \mathrm{M}\left(\mathrm{u}, \ \mathrm{u}, \frac{t}{k}\right)}{M\left(z, \ u, \ \frac{t}{k}\right)}\right\}$$

This implies that either $\lambda\ (z, u, \frac{t}{k}) = 1$ or $\lambda\ (z, u, \frac{t}{k}) = \mathrm{M}\ (z, u, \frac{t}{k})$

Using it in previous equation, we get z = u.

Thus, u is a unique fixed point of p. This completes the proof of Theorem 2.1

COROLLARY 2.1. Let (X, M, *) be a complete fuzzy metric space and p :X → X be a mapping satisfying

$\mathrm{M}\ (px, py, kt) \geq \lambda\ (x, y, t)$

where,

$\lambda\ (x, y, t) = \left\{M\ (x, y, t) + M\ (x, px, t) + M(y, py, t) + \frac{\mathrm{M}(x, px, t) * \mathrm{M}(y, py, t)}{M(x, y, t)}\right\}$

for all x, y ∈ X, and k ∈ (0, 1). Then p has a unique fixed point . The proof of the result follows immediately from Theorem 2.1 by taking ξ(t) = t.

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3.Application.

Theorem 3.1

In this section, we give an application related to our result. Let us define ψ : [0, ∞] → [0, ∞], as

$\Psi(t) = \int_0^t \Psi(t)dt$ ∀ t > 0, be a non-decreasing and continuous function. Moreover, for each ϵ > 0, Ψ(t) > 0 and Ψ(t) = 0 iff t = 0.

Let (X, M, *) be a complete fuzzy metric space and p : X → X be a mapping satisfying

$$\int_0^{M(px,py,kt)} \Psi(t)\,dt > \xi\left\{\int_0^{\lambda\,(x,y,t)} \Psi(t)\ dt\right\}$$

Where,

$\lambda\,(x,y,t) =$

$$\left\{M(x,y,t) + M(x,px,t) + M(y,py,t) + \frac{M(x,px,t) * M(y,\ py,t)}{M(x,y,\ t)}\right\}$$

for all x, y ∈ X, ϕ ∈ Ψ, ξ ∈ ∅ and k ∈ (0, 1), Then p has a unique fixed point.

Proof: By taking Ψ(t) = 1 and applying Theorem 3.1, we obtain the result.

Conclusion

This study establishes the existence and uniqueness of a fixed point in fuzzy metric spaces by formulating an appropriate contractive-type condition governed by a control function. The introduced framework relaxes classical contraction assumptions and accommodates a broader class of nonlinear mappings within fuzzy environments. By integrating control functions with fuzzy metrics, the convergence of iterative sequences toward the fixed point is ensured, enhancing the robustness of the results. The proposed theorem not only unifies and extends several known fixed point results in fuzzy metric spaces but also provides a flexible analytical tool. These findings have potential applications in decision theory, optimization, and fuzzy differential equations.

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