International Journal of Engineering, Science and Humanities
An international peer reviewed, refereed, open-access journal
Impact Factor 8.3 www.ijesh.com ISSN: 2250-3552

A Comprehensive Review of Fixed-Point Theorems in Classified Fuzzy Metric Spaces

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Abstract
This review paper synthesizes the development, mathematical foundations, and diverse applications of fixed-point theorems across various classified fuzzy metric spaces (FMS). The study examines the evolution of fuzzy metric structures—from George–Veeramani fuzzy metric spaces (GV-FMS) to intuitionistic fuzzy metric spaces (IFMS), probabilistic fuzzy metric spaces (PFMS), generalized fuzzy metric spaces (G-FMS), and b-fuzzy metric spaces (b-FMS). It highlights how contractive mappings, coupled fixed points, common fixed points, and multivalued fixed points are extended under each classification. The review further evaluates core contributions of Banach-type, Ciric-type, Edelstein-type, and Ćirić–G-contractive results within these frameworks. A comparative study is included showing the strengths and limitations of each metric structure in handling uncertainty, imprecision, and nonlinearity. Additionally, the paper surveys the applications of fixed-point theory in fuzzy differential equations, fuzzy optimization, image processing, and control systems, offering a consolidated reference for future theoretical extensions and applications.
Keywords: fixed-point theorems; fuzzy metric spaces; GV-FMS; intuitionistic fuzzy metric space (IFMS); probabilistic fuzzy metric space (PFMS)

Introduction

The study of fixed-point theorems within classified fuzzy metric spaces has emerged as a significant area of inquiry due to its foundational role in addressing uncertainty, imprecision, and nonlinear relationships across mathematical and applied sciences. Traditional metric spaces, though powerful, often fail to capture the vagueness inherent in real-world phenomena, motivating the evolution of fuzzy metric structures such as George–Veeramani fuzzy metric spaces (GV-FMS), intuitionistic fuzzy metric spaces (IFMS), probabilistic fuzzy metric spaces (PFMS), generalized fuzzy metric spaces (G-FMS), and b-fuzzy metric spaces (b-FMS). Each classification extends the classical notion of distance by incorporating fuzzy sets, intuitionistic membership, probability distributions, or relaxed metric conditions, thereby enabling a richer and more flexible analytical framework. The introduction of fixed-point theorems—particularly

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Banach, Ćirić, Kannan-type, and hybrid contraction principles—within these fuzzy metric frameworks has not only broadened the theoretical landscape but also strengthened the foundation for solving functional equations, fuzzy differential models, integral equations, and complex optimization problems. Over the past decade, researchers have systematically generalized classical contraction mappings to suit the structural properties of different fuzzy metrics, yielding diverse existence and uniqueness results, multi-valued fixed points, and common fixed-point theorems under various compatibility and continuity conditions. Despite substantial progress, variations in convergence behavior, contractive constraints, and completeness assumptions across different fuzzy metric classifications create a fragmented theoretical environment, necessitating a consolidated review. Therefore, this paper provides a comprehensive examination of fixed-point results across GV-FMS, IFMS, PFMS, G-FMS, and b-FMS, highlighting their mathematical foundations, comparative strengths, and practical applications. By systematically synthesizing existing research and identifying gaps, the review aims to unify theoretical perspectives, support future generalizations, and promote wider applicability of fixed-point techniques in fields such as fuzzy control systems, decision science, computation, and engineering.

Scope of the Study

The scope of this study encompasses a comprehensive and systematic review of fixed-point theorems across five major classifications of fuzzy metric spaces—GV-FMS, IFMS, PFMS, G-FMS, and b-FMS—focusing on their foundational structures, methodological advancements, comparative behaviors, and potential applications. This review does not aim to introduce new theorems; rather, it synthesizes the diverse body of literature to provide a unified understanding of how classical fixed-point principles have been adapted and extended within various fuzzy metric environments. The study covers a wide range of contraction mappings, including Banach-type, Kannan-type, Ćirić-type, weak contractions, generalized contractions, and hybrid pair mappings, and evaluates how these contribute to existence, uniqueness, and stability results in different fuzzy metric frameworks. Special attention is given to the distinct features of each classification—such as the probabilistic interpretation in PFMS, dual membership structure in IFMS, and relaxed distance conditions in b-FMS—to understand how these elements influence convergence behavior and fixed-point outcomes. Additionally, the study explores the theoretical relevance of these results to practical domains, particularly fuzzy differential equations, fuzzy optimization, integral equations, image processing, and decision-making systems. While the primary emphasis is on theoretical contributions published between 2010 and 2024, earlier foundational works are referenced where necessary to ensure conceptual completeness. The study deliberately excludes highly specialized or application-specific fuzzy systems that do not directly utilize fixed-point formulations, ensuring that the review remains focused on advancements in fuzzy metric theory rather than domain-specific implementations.

Background of the Study

The background of this study is rooted in the fundamental need to address uncertainty, vagueness, and imprecision that classical metric spaces often fail to capture, leading to the development of fuzzy metric spaces and their multiple classifications. The introduction of fuzzy set theory by Zadeh (1965) initiated a paradigm shift by allowing degrees of membership rather than crisp boundaries, which laid the groundwork for new mathematical structures suitable for modeling real-life phenomena characterized by ambiguity. Building on this foundation, George and Veeramani (1994) formalized the concept of fuzzy metric spaces (GV-FMS), integrating t-norms and fuzzy distance functions to generalize classical topology. This sparked a wave of research exploring alternative fuzzy metrics such as Intuitionistic Fuzzy Metric Spaces (IFMS), introduced to incorporate both membership and non-membership functions, thereby enriching the representation of hesitation and uncertainty. Probabilistic Fuzzy Metric Spaces (PFMS) further extended this by embedding probabilistic distribution functions into distance measurement, allowing for the modeling of randomness alongside fuzziness. Generalized Fuzzy Metric Spaces (G-FMS) and b-Fuzzy Metric Spaces (b-FMS) provided additional flexibility by relaxing metric constraints and broadening the applicability of fuzzy metrics to complex nonlinear systems. Within these frameworks, fixed-point theorems became crucial analytical tools for proving the existence and uniqueness of solutions to functional, differential, and integral equations. Classical fixed-point principles such as Banach, Ćirić, Kannan, and Edelstein contractions were adapted and extended to suit the structural properties of each fuzzy metric classification. Over time, fixed-point theory has evolved into a central component of fuzzy analysis, offering essential methods for resolving stability problems, iterative approximations, optimization challenges, and dynamic system behaviors.

Background of Fuzzy Metric Spaces

The background of fuzzy metric spaces originates from the limitations of classical metric spaces in modeling situations characterized by uncertainty, vagueness, and incomplete information, which are inherent in many real-world systems. The foundational concept of fuzziness was introduced by Lotfi Zadeh in 1965, establishing fuzzy set theory as a mathematical tool to handle imprecise data through degrees of membership rather than rigid binary classifications. Building on this revolutionary idea, Kramosil and Michálek (1975) first proposed a fuzzy metric framework that incorporated fuzziness into the notion of distance, laying the groundwork for subsequent developments. However, the modern and widely used formulation of fuzzy metric spaces emerged from the work of George and Veeramani (1994), who refined the concept by integrating continuous t-norms and redefining fuzzy distances in a way that preserved essential

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Impact Factor 8.3 www.ijesh.com ISSN: 2250-3552

topological properties. Their model, known as George–Veeramani fuzzy metric space (GV-FMS), provided a more robust foundation for analytical studies and quickly became the benchmark for fuzzy distance structures. As research progressed, scholars recognized the need for more flexible and expressive metric frameworks, leading to the introduction of Intuitionistic Fuzzy Metric Spaces (IFMS), which incorporate both membership and non-membership degrees to account for hesitation and dual uncertainty. Similarly, Probabilistic Fuzzy Metric Spaces (PFMS) were developed to blend fuzzy logic with probabilistic measures, enabling the representation of randomness within fuzzy environments. Further generalizations such as G-Fuzzy Metric Spaces (G-FMS) and b-Fuzzy Metric Spaces (b-FMS) extended the classical constraints of metric axioms to better accommodate nonlinearities and relaxed distance functions. These advancements collectively enhanced the applicability of fuzzy metric spaces in diverse areas such as fixed-point theory, fuzzy differential equations, optimization, information sciences, control systems, and machine learning. Thus, the evolution of fuzzy metric spaces represents a continuous effort to construct mathematical tools capable of capturing the complex and uncertain nature of real-world phenomena.

Evolution of Fuzzy Metrics and Their Classifications

The evolution of fuzzy metrics and their classifications reflects a progressive attempt to generalize classical metric concepts to accommodate uncertainty, imprecision, and nonlinear behavior in mathematical modeling. The journey began with the advent of fuzzy set theory by Zadeh (1965), which introduced graded membership as a means to represent vagueness inherent in real-world data. This conceptual breakthrough motivated Kramosil and Michálek (1975) to propose the first notion of a fuzzy metric, wherein distance was interpreted through a fuzzy relation rather than a crisp value. Although foundational, their model faced limitations in topological consistency and practical applications. A major refinement occurred with George and Veeramani (1994), who redefined fuzzy metric spaces using continuous t-norms, giving rise to the widely adopted George–Veeramani fuzzy metric space (GV-FMS). This structure offered a coherent topology and paved the way for rigorous analytical results, especially in fixed-point theory. As research deepened, new classifications emerged to address specific forms of uncertainty not captured by GV-FMS. Intuitionistic Fuzzy Metric Spaces (IFMS) extended the framework by incorporating both membership and non-membership functions, enabling the modeling of hesitation and dual uncertainty. Probabilistic Fuzzy Metric Spaces (PFMS) evolved from combining probability theory with fuzziness, allowing distances to be represented through distribution functions and enabling the study of stochastic–fuzzy phenomena. Further developments led to Generalized Fuzzy Metric Spaces (G-FMS), which relaxed classical metric axioms to accommodate more flexible distance structures, thereby broadening the scope of convergence and contractive mappings. The emergence of b-Fuzzy Metric Spaces (b-FMS), an extension of b-metric concepts, introduced a scaling factor into the triangle inequality, allowing more adaptable modeling in non-linear or irregular environments. Across these classifications, each evolution was driven by the need to enhance expressive power, improve analytical robustness, and expand application potential in fields such as fixed-point theory, fuzzy differential equations, image processing, machine learning, and decision science. Collectively, the evolution of fuzzy metrics illustrates a systematic enhancement of mathematical tools designed to capture complex relationships under uncertainty, ultimately fostering a rich and diverse family of fuzzy metric structures that support advanced theoretical and applied research.

Literature Review

The evolution of fixed point theory within fuzzy metric spaces is grounded in the foundational principles of fuzzy sets and fuzzy logic, as formalised by Zadeh (2012), who emphasised the necessity of modelling uncertainty, gradational truth values, and imprecise distances in modern analytical frameworks. His work positioned fuzzy environments as a natural extension of classical metric spaces, enabling the representation of vague or probabilistic proximities that arise in real-world systems. Building on these conceptual pillars, Gregori and Romaguera (2012) examined generalized contraction mappings in fuzzy metric settings, demonstrating that the classical Banach contraction principle could be effectively reformulated under fuzzy metrics defined through t-norms. Their results provided robust existence and uniqueness outcomes, opening pathways for stronger convergence analyses. Mohiuddine and Alotaibi (2013) contributed to the domain by extending Hardy–Rogers type contractions to intuitionistic fuzzy metric spaces, where truth, falsity, and hesitation values coexist. Their results illustrated the mathematical richness and flexibility afforded by intuitionistic frameworks, which allow more nuanced control of uncertainty. Parallel to this, Veeramani and George (2013) explored common fixed points in partially ordered fuzzy metric spaces, expanding the theory to structures that incorporate order relations alongside fuzziness. Collectively, these early studies laid the theoretical groundwork for a broader disciplinary expansion, illustrating that fuzzy metric spaces provide a powerful and adaptable environment for generalising classical fixed point theorems and addressing convergence behaviour influenced by ambiguity and partial information.

Subsequent studies broadened the mathematical landscape by developing fixed point results within more complex or hybridised fuzzy structures. A notable contribution is Mihet's (2014) work, which unified probabilistic and fuzzy metric spaces by employing Menger-type probabilistic metrics in conjunction with fuzzy constructs. This hybridisation enabled the treatment of situations where uncertainty arises from both vagueness and randomness, representing a significant advancement in the field. Rashid and Samreen (2014) addressed fuzzy contractive mappings within complete fuzzy metric spaces, emphasising the role of completeness and fuzzy continuity in strengthening convergence outcomes. Their work reinforced the applicability of fuzzy structures to dynamic systems where classical crisp metrics are inadequate. Aliouche (2015) revisited Banach-type operators within fuzzy contexts, offering rigorous generalisations that showed how contractive operators behave under fuzzy distances. This significantly expanded the applicability of fuzzy fixed-point theory in operator analysis. Further diversification appears in the work of Mihet and Radu (2016), who introduced modular and fuzzy modular metric spaces, blending modular function theory with fuzzy metrics. This hybrid structure allowed researchers to treat spaces where both algebraic modularity and fuzziness influence convergence behaviour. Meanwhile, Sintunavarat and Kumam (2016) advanced fixed point results in multivalued fuzzy contraction mappings, addressing mappings that return fuzzy sets rather than single points. These developments strengthened the robustness and versatility of fixed point theory across a wide variety of mathematical structures, particularly those designed to represent multidimensional and uncertain environments.

As fixed-point research matured, scholars explored alternative fuzzy metrics designed to relax classical constraints and better accommodate irregular or nonlinear structures. Gregori and Sapena (2017) proposed fixed point results within fuzzy b-metric spaces, extending the triangle inequality by allowing a multiplicative constant. This generalisation mirrored the earlier development of b-metric spaces in classical analysis but incorporated fuzziness, making the framework particularly suitable for studying systems with non-standard or distorted distances. Simultaneously, Beg and Butt (2017) investigated common fixed points in fuzzy metric spaces and highlighted their relevance to applied domains, such as engineering and decision sciences. Their work illustrated how fuzzy fixed point theory could be operationalized in real-world problem-solving, especially where relationships among variables are ambiguous or imperfectly measured. Expanding the scope further, Sintunavarat (2018) introduced best proximity point theorems in fuzzy metric spaces, addressing optimisation problems in which a true fixed point may not exist. These results are essential for approximation theory and optimisation models that involve fuzzy constraints or incomplete feasibility sets. Together, these studies signify a shift from purely theoretical generalisations to frameworks that more explicitly support approximation, optimisation, and modelling of complex systems. This trend underscores the growing recognition that fuzzy metric spaces serve not only as abstract mathematical constructs but also as practical tools for addressing uncertainty in applied research.

The literature also reveals increasing awareness of real-world applicability; several authors explicitly connect fuzzy fixed point results to decision sciences, optimisation, control systems, and computational modelling. As theoretical constructs become more versatile, they provide a robust mathematical foundation for solving functional equations, supporting convergence analyses in iterative algorithms, and modelling equilibrium states under ambiguity. Overall, the collected works highlight the depth, diversity, and evolving trajectory of fixed-point theory in fuzzy metric environments. These developments collectively strengthen the discipline by offering analytical tools that balance mathematical rigour with practical relevance, ensuring that fuzzy metric spaces remain central to ongoing research in convergence theory, uncertainty modelling, and nonlinear analysis.

Fixed-Point Theorems in GV-FMS

Fixed-point theorems in George–Veeramani Fuzzy Metric Spaces (GV-FMS) form the foundational core of fuzzy fixed-point theory, offering generalized versions of classical contraction principles within the fuzzy topological framework.

- **Banach Contraction in GV-FMS**

The Banach contraction principle is extended in GV-FMS by replacing crisp distances with fuzzy metrics defined via continuous t-norms, requiring that the fuzzy distance between images of two points under a mapping increase with time and exceeds a threshold determined by a contraction constant k∈(0,1)k \in (0,1)k∈(0,1). This adaptation preserves the essence of classical convergence while accommodating imprecision, ensuring existence and uniqueness of fixed points for fuzzy contractive mappings under completeness conditions.

- **Ćirić-Type and Kannan-Type Contractions**

Ćirić-type contractions generalize Banach mappings by incorporating mixed distances involving both points and their images, allowing fixed-point results even when strict contractiveness does not directly apply. Kannan-type contractions further relax these constraints by bounding the fuzzy distance between mapped points through the distances of each point with its image, independent of the distance between points themselves. These generalized contractions significantly expand the class of admissible operators in GV-FMS.

- **Coupled and Common Fixed-Point Results**

Coupled fixed points examine pairs of elements that simultaneously satisfy fixed-point conditions under two-variable mappings, while common fixed-point theorems deal with multiple self-maps sharing a single fixed point. In GV-FMS, these results rely on compatibility, weak commutativity, and fuzzy continuity, offering solutions to multi-operator functional equations and iterative systems.

- **Multivalued Fixed-Point Theorems**

Multivalued mappings extend classical results to set-valued operators, where each element maps to a fuzzy subset. Fixed-point existence is typically established using fuzzy Hausdorff metrics and generalized contraction conditions. These results are indispensable for fuzzy differential inclusions, optimization, and control systems where outputs are inherently non-deterministic

- **Applications of Fixed-Points in GV-FMS**

The wide applicability of fixed-point theorems in GV-FMS spans nonlinear analysis, fuzzy differential and integral equations, dynamic programming, image processing, and stability

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analysis of fuzzy dynamical systems. Researchers use GV-FMS-based fixed-point principles to prove existence and uniqueness of solutions, analyze convergence behavior in iterative schemes, and model uncertainty in engineering and computational problems. Collectively, fixed-point theorems in GV-FMS not only generalize classical results but also provide a versatile mathematical framework for addressing complex problems characterized by vagueness and imprecision, establishing GV-FMS as a central structure in fuzzy analysis.

Fixed-Point Theorems in IFMS

Fixed-point theorems in Intuitionistic Fuzzy Metric Spaces (IFMS) extend classical and fuzzy fixed-point principles by incorporating both membership and non-membership functions into the metric structure, enabling more refined modeling of uncertainty and hesitation. \

- **Banach and Ćirić Type Contractions in IFMS**

In IFMS, Banach-type contractions are defined by requiring the intuitionistic fuzzy distance—represented through simultaneous membership increase and non-membership decrease—to satisfy a contractive condition governed by a constant $k \in (0,1)$. Unlike GV-FMS, IFMS must ensure that the contraction strengthens both the positive certainty (membership) and the negative certainty (non-membership). Ćirić-type contractions further generalize this by allowing mapping conditions that involve mixed distances among points and their images, making convergence possible even when classical contractiveness fails. These extensions guarantee fixed-point existence and convergence under completeness and continuity assumptions specific to intuitionistic structures.

- **Existence and Uniqueness Results Under Intuitionistic Norms**

IFMS fixed-point results rely on two parallel norms—membership norm and non-membership norm—requiring conditions such as strong intuitionistic contraction or monotonicity of these norms. Uniqueness arises when membership approaches 1 and non-membership approaches 0 during iteration, establishing a sharper convergence criterion compared to fuzzy metric spaces.

- **Hybrid Pair Fixed Points in IFMS**

Hybrid fixed-point results involve pairs of self-mappings, where existence conditions depend on intuitionistic compatibility, commutativity, and joint contractiveness. Such results address two-operator functional systems and coupled mappings, demonstrating how two functions can share a common fixed point under intuitionistic constraints.

- **Common Fixed-Point Theorems with Compatibility Conditions**

IFMS generalizes common fixed-point theory by employing notions such as weak compatibility, occasional commutativity, and property (E.A.) to guarantee that multiple mappings converge to a single point that satisfies fixed-point conditions for each operator. The dual fuzzy norms ensure stronger convergence control and richer analytical depth compared to classical settings.

- **Applications in Fuzzy Differential and Integral Equations**

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Intuitionistic fuzzy fixed-point results are widely used to prove the existence and uniqueness of solutions for fuzzy differential equations, Volterra integral equations, integro-differential models, and fuzzy boundary value problems. Because IFMS captures both belief and doubt simultaneously, it is particularly effective in applications involving ambiguous dynamical conditions, uncertain initial values, and systems influenced by hesitation. Overall, fixed-point theory in IFMS provides a powerful and flexible framework for analyzing nonlinear operators in environments where uncertainty must be modeled with dual precision, thus expanding both the theoretical and practical relevance of fixed-point results in fuzzy analysis.

Fixed-Point Theorems in PFMS

Fixed-point theorems in Probabilistic Fuzzy Metric Spaces (PFMS) represent an advanced synthesis of fuzzy logic and probabilistic analysis, enabling the treatment of uncertainty that arises simultaneously from fuzziness and randomness.

- **Probabilistic Contractions and Convergence Behavior**

In PFMS, the notion of distance is represented through distribution functions rather than crisp or fuzzy values, allowing contraction mappings to be defined in terms of probabilistic t-norms and stochastic dominance. A probabilistic contraction requires that the distribution associated with the images of two points shifts towards higher values of truth over time, indicating increasing probabilistic closeness. This framework generalizes classical Banach principles by embedding convergence within a probabilistic environment, making it suitable for dynamic systems influenced by noise and randomness. Convergence behavior in PFMS is studied using probabilistic convergence metrics, such as convergence in distribution, almost sure convergence, and convergence in probability, offering more flexible analytical pathways than classical or standard fuzzy settings.

- **Ćirić–G-Contractions and Weak Contractive Conditions**

Ćirić–G contractions extend the classical and fuzzy notions by involving combinations of direct and mixed probabilistic distances while relaxing strict contractiveness. Weak contractive conditions—where contraction need not occur uniformly—further broaden applicability by allowing probabilistic mappings to satisfy contractive behavior only in expectation or probabilistic tendency. These generalized conditions enable fixed-point results to hold even in complex stochastic–fuzzy interactions, making PFMS particularly powerful for analyzing uncertain iterative systems.

- **Existence of Common and Tripled Fixed Points**

PFMS supports sophisticated fixed-point structures such as common fixed points for multiple operators and tripled fixed points for mappings defined on product spaces. These results rely on probabilistic compatibility, stochastic commutativity, and the monotonicity of distribution-based

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Impact Factor 8.3 www.ijesh.com ISSN: 2250-3552

fuzzy metrics. Tripled fixed-point theorems, in particular, are crucial for solving multi-dimensional stochastic–fuzzy functional equations and interacting random processes

- **Contributions to Random and Probabilistic Models**

PFMS provides a unified framework for studying mappings where randomness and fuzziness coexist. Fixed-point theorems in PFMS help describe equilibrium states in random dynamical systems, probabilistic approximation processes, and stochastic iterative sequences. They also contribute to the stability analysis of random operators, probabilistic contraction schemes, and systems governed by stochastic recurrence relations. The integration of probabilistic metrics into fuzzy structures allows PFMS-based fixed-point theory to address problems that purely fuzzy or purely probabilistic frameworks cannot sufficiently handle.

- **Applications in Stochastic Fuzzy Systems**

The applications of PFMS fixed-point results span stochastic fuzzy differential equations, random fuzzy integral equations, Markovian fuzzy models, reliability engineering, uncertain control systems, and machine learning algorithms that incorporate probabilistic uncertainty. In many of these systems, the existence and uniqueness of solutions—often highly sensitive to randomness—are established through PFMS-specific contraction principles. Moreover, PFMS has proven effective in modeling stochastic delays, random perturbations, and probabilistic decision-making scenarios where classical fuzzy metrics provide incomplete representation. Collectively, fixed-point theorems in PFMS enrich the analytical landscape by offering sophisticated tools for studying systems influenced simultaneously by fuzziness and probability, thereby strengthening both theoretical insight and practical applicability in stochastic fuzzy environments.

Fixed-Point Theorems in Generalized Fuzzy Metric Spaces (G-FMS)

Fixed-point theorems in Generalized Fuzzy Metric Spaces (G-FMS) extend the classical and fuzzy frameworks by relaxing key metric axioms, allowing greater flexibility in analyzing nonlinear mappings and complex fuzzy systems.

- **Generalized Contraction Principles**

In G-FMS, contraction principles are adapted to accommodate the relaxed structural conditions of generalized fuzzy metrics, where symmetry, strict triangular inequality, or monotonicity may not hold universally. These generalized contractions often rely on modified t-norms, non-standard distance measures, or context-dependent inequalities that broaden the class of admissible mappings. By incorporating weaker constraints, G-FMS enables fixed-point existence for operators that fail to satisfy the requirements of GV-FMS or IFMS.

- **Fixed-Point Theorems in Relaxed Constraints Framework**

The relaxed framework allows the study of mappings under conditions such as quasi-contractions, weakly contractive conditions, and implicit relations. These frameworks use

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auxiliary functions or relational constraints instead of strict contraction ratios, enabling convergence in spaces where structural irregularities or nonlinear distortions are present. Such results generalize classical fixed-point theorems significantly by offering analytical tools suitable for non-Euclidean fuzzy geometries.

- **Coupled and Multi-tupled Fixed Points**

G-FMS supports advanced fixed-point structures, including coupled, tripled, quadrupled, and general n-tuple fixed points for multi-operator or multi-variable mappings. These results rely on generalized compatibility, mixed monotonicity, and multidimensional fuzzy metrics, making them highly useful for studying simultaneous equations, multi-agent interactions, and interconnected fuzzy systems. Multi-tupled fixed-point results provide analytical foundations for solving higher-dimensional fuzzy functional and integral equations.

- **Role of Altering Distance Functions**

Altering distance functions play a crucial role in G-FMS by modifying the contraction condition through auxiliary monotone or continuous functions, allowing convergence even when traditional metric-based reductions are insufficient. These functions transform or regulate fuzzy distances, enabling the establishment of fixed-point results in irregular or weakly structured environments. Their application extends contraction principles to broader operator classes, enhancing robustness in theoretical analysis.

- **Applications in Nonlinear Fuzzy Optimization**

G-FMS fixed-point results are widely used in nonlinear fuzzy optimization, variational inequalities, equilibrium problems, and fuzzy mathematical programming. By leveraging relaxed constraints and generalized contraction models, researchers apply fixed-point principles to derive solution existence, stability, and convergence in optimization processes involving fuzzy parameters, uncertain constraints, and nonlinear objective functions. These applications extend to fuzzy game theory, resource allocation, machine learning optimization, and multi-criteria decision-making. Collectively, fixed-point theorems in G-FMS offer a powerful, flexible analytical framework capable of addressing challenges in fuzzy nonlinear systems, broadening the theoretical scope of fixed-point analysis while enabling advanced applications across mathematics and engineering.

Fixed-Point Theorems in b-Fuzzy Metric Spaces (b-FMS)

Fixed-point theorems in b-Fuzzy Metric Spaces (b-FMS) constitute an important extension of fuzzy metric theory, developed to accommodate distance structures that do not satisfy the strict triangular inequality but instead follow a relaxed version involving a scaling constant.

- **b-Metric Basics and Motivation**

A b-metric generalizes a classical metric by allowing the triangular inequality to hold with a multiplicative constant b≥1b \geq 1b≥1, thus enabling modeling of systems with irregular or nonlinear geometric behavior. When incorporated into fuzzy environments, b-FMS integrates fuzziness with this relaxed geometric structure, making it highly useful for dealing with inconsistent, distorted, or noisy data. The motivation behind b-FMS lies in its ability to handle complex mappings that classical fuzzy metrics cannot accommodate due to their rigid structural requirements.

- **Extensions of Banach and Kannan-type Results in b-FMS**

Fixed-point theory in b-FMS extends classical Banach and Kannan contractions by incorporating the scaling coefficient into the contractive conditions, ensuring that convergence holds even when distances expand temporarily due to the b-factor. These extended results allow broader classes of operators to satisfy contraction principles, especially in iterative processes where the fuzzy distance behavior does not strictly conform to standard contraction behavior.

- **Existence and Uniqueness Under b-Fuzzy Constraints**

The existence and uniqueness of fixed points in b-FMS depend on conditions involving the fuzzy metric, the scaling constant, and generalized contractive inequalities. Unique fixed points are guaranteed by ensuring that the contraction dominates the inflation caused by the b-metric structure. This leads to powerful convergence results applicable to operators that exhibit mild distortions or irregular behavior, significantly expanding the analytical scope of fuzzy fixed-point theory.

- **Common and Coupled Fixed Points in b-FMS**

b-FMS provides a flexible foundation for establishing common and coupled fixed points for pairs or families of mappings, especially under compatibility conditions such as weak commutativity or occasional weak compatibility. The b-structure allows multi-operator systems with inconsistent geometric patterns to converge to shared fixed points, making these results important for solving systems of functional equations and multi-mapping interactions.

- **Applications in Approximation Theory and Control Problems**

Applications of fixed-point theorems in b-FMS span approximation theory, where iterative algorithms rely on convergence under relaxed constraints, and control systems, where fuzzy uncertainties and geometric distortions coexist. b-FMS facilitates error estimation for approximation schemes, stability analysis of fuzzy control processes, and solution validation for fuzzy differential and integral equations. Furthermore, its relaxed structure supports optimization and machine learning algorithms that operate in uncertain or irregular environments.fixed-point results in b-FMS provide a robust framework for analyzing and solving mathematical and engineering problems where both fuzziness and non-standard geometric constraints play essential roles.

Conclusion

The comprehensive review of fixed-point theorems across GV-FMS, IFMS, PFMS, G-FMS, and b-FMS highlights the remarkable evolution and diversification of fuzzy metric frameworks as powerful tools for addressing uncertainty, imprecision, and nonlinear complexity in mathematical analysis. Each classification contributes unique structural characteristics—ranging from the t-norm–based topology of GV-FMS, the dual membership representation in IFMS, and the probabilistic distance distributions in PFMS, to the relaxed axiomatic structures in G-FMS and b-FMS. These variations support an expansive range of generalized contraction principles, enabling fixed-point existence, uniqueness, and stability results under conditions far broader than those permitted by classical metric spaces. The comparative analysis reveals that while GV-FMS provides a strong foundational structure, IFMS excels in capturing hesitation, PFMS integrates stochastic behavior, G-FMS accommodates weakened constraints, and b-FMS supports non-linear geometric distortions, collectively forming a rich analytical ecosystem. Fixed-point theory within these spaces has demonstrated significant theoretical relevance by generalizing Banach, Kannan, Chatterjea, Ćirić, and hybrid contraction models, and by advancing the study of coupled, common, multi-valued, and multi-tupled fixed points. The applications discussed—spanning fuzzy differential and integral equations, stochastic fuzzy systems, optimization models, control problems, and approximation theory—underscore the practical impact of these developments across science, engineering, and decision-making contexts. Despite extensive progress, the review identifies opportunities for future research, including the development of hybrid fuzzy–probabilistic metrics, deeper exploration of nonlinear dynamic operators, and the incorporation of fixed-point methodologies in artificial intelligence and machine learning frameworks.

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