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Advanced Numerical Methods for Generalized Caputo-Type Fractional-Order Differential Systems: Theory, Implementation, and Chaotic Applications

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Abstract
Fractional-order differential systems offer a robust framework for modeling real-world phenomena with memory and hereditary characteristics, yet their numerical solution remains challenging. This study introduces two advanced numerical methods the adaptive predictor-corrector (P-C) and the generalized Laplace decomposition method (TρDM) to efficiently solve equations involving generalized Caputo-type derivatives. The proposed methods are tested on the Abel differential equation and the four-dimensional Chen system, capturing both regular and chaotic dynamics. Numerical results reveal that the new algorithms achieve high accuracy, computational efficiency, and flexibility, outperforming classical approaches such as the Adams-Bashforth-Moulton (ABM) method. The study's findings demonstrate that these methods are reliable and effective, providing powerful tools for scientists and engineers tackling complex nonlinear fractional-order systems.

Keywords: Fractional differential equations, Caputo derivative, adaptive predictor-corrector method, generalized Laplace decomposition, chaos, Abel equation, Chen system, numerical simulation, computational efficiency, memory effects

1. Introduction

Fractional-order differential equations (FDEs) have revolutionized mathematical modeling in science and engineering by enabling the accurate representation of memory effects, hereditary phenomena, and nonlocal dynamics that integer-order models cannot capture. These systems have found wide-ranging applications in physics, biology, economics, finance, and control theory. The rise of generalized fractional operators, notably the Caputo-type derivative, has further enhanced the flexibility and applicability of FDEs to real-world problems, including those exhibiting chaotic or hyperchaotic behavior. Despite their theoretical appeal, the practical solution of FDEs remains a computational challenge. Established numerical methods, such as the

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Adams-Bashforth-Moulton (ABM) algorithm, often encounter issues related to stability, convergence, or efficiency, especially for systems with high dimensionality or complex boundary conditions. Recent advances have introduced more robust and adaptive algorithms including predictor-corrector schemes and Laplace decomposition techniques that promise improved performance for nonlinear, memory-dependent systems.

This paper is motivated by the need to develop and comparatively assess advanced numerical approaches for generalized Caputo-type fractional-order systems. We focus on two methods: an adaptive predictor-corrector (P-C) algorithm and the generalized Laplace decomposition method (TρDM), both designed for efficiency, accuracy, and adaptability. Through rigorous computational experiments on benchmark problems—the Abel equation and the 4D Chen system we demonstrate the superiority and practical relevance of these techniques for capturing intricate dynamical phenomena.

2. Scope of the Study

This research aims to advance the numerical analysis of fractional-order differential systems involving generalized Caputo-type derivatives. The scope includes:

- Theoretical formulation of generalized Caputo-type derivatives and their properties.
- Development and implementation of the adaptive predictor-corrector (P-C) and generalized Laplace decomposition (TρDM) methods.
- Application to both canonical nonlinear equations (Abel) and high-dimensional chaotic systems (the fractional Chen system).
- Comparative analysis with the Adams-Bashforth-Moulton method in terms of accuracy, efficiency, and capability to detect chaotic dynamics.
- Evaluation of numerical stability, convergence, error, and computational cost across varying fractional orders and parameter settings.

3. Need of the Study

There is an urgent need for reliable and efficient numerical methods tailored to fractional-order differential systems, as these are increasingly used to model phenomena with memory, nonlocality, and complexity. Existing tools often face limitations in accuracy, convergence, and computational overhead, particularly for highly nonlinear or chaotic systems. The ability to flexibly vary the fractional order and initial conditions is crucial for exploring a wide range of real-world behaviors, from anomalous diffusion to chaos. This study addresses these gaps by proposing and thoroughly testing new algorithms that enhance both theoretical understanding and practical computation in this rapidly growing field.

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4. Review of Literature

Podlubny, I. (2015) Developed adaptive numerical schemes for Caputo and Riemann-Liouville fractional derivatives, focusing on stability and long-term integration. His work laid the groundwork for robust computational frameworks in fractional calculus. Diethelm, K. (2016) Proposed improved predictor-corrector methods, providing detailed error analyses for fractional differential equations. These advancements helped enhance both the reliability and efficiency of numerical solutions. Li, C., & Zeng, F. (2017) Reviewed generalized fractional operators and highlighted their effectiveness in modeling nonlinear systems and chaotic dynamics, broadening the scope of fractional calculus in applied mathematics. Garrappa, R. (2018) Presented efficient algorithms for simulating fractional-order chaotic systems, including cases with multidimensionality and non-Lipschitz dynamics, greatly expanding practical applications. Wang, J. (2019) Demonstrated numerical approaches for variable-order and multi-term fractional systems, with specific applications in control engineering and signal processing, thus illustrating versatility in real-world contexts. Baleanu, D. (2020) Investigated memory and hereditary effects in complex systems using generalized Caputo derivatives, offering new insights into the modeling of systems with long-term dependencies. Chen, Y. (2021) Developed Laplace-based decomposition methods for fractional nonlinear oscillators, comparing their performance with classical techniques and demonstrating improvements in computational efficiency. Atangana, A. (2022) Extended fractional calculus to multi-dimensional and stochastic systems, introducing hybrid analytical-numerical methods that improved solution accuracy for complex models. Zhang, H. (2023) Addressed convergence and computational cost issues in large-scale simulations, contributing strategies to make generalized fractional-order equation solvers more scalable and practical. Li, Z. (2024) Explored integrating machine learning with numerical solvers for fractional-order systems, enhancing predictive capabilities and automating parameter selection in complex simulations. Singh, R. (2025) Provided a unified comparative framework for adaptive predictor-corrector and Laplace decomposition approaches, especially in the context of chaotic fractional systems, guiding method selection for future research.

5. Data Analysis and Results

This section presents the comprehensive results obtained from the theoretical formulation, numerical experimentation, and computational implementation of advanced numerical methods for generalized Caputo-type fractional-order differential systems. The core focus lies in evaluating the efficiency, accuracy, and robustness of the adaptive predictor-corrector (P-C) method and the generalized Laplace decomposition method (TρDM) in solving both canonical and high-dimensional nonlinear systems. Through a series of benchmark problems including the Abel differential equation and the four-dimensional Chen system we systematically analyze

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convergence, error, stability, and the capability to capture complex dynamics such as chaos. The discussion begins with essential definitions and mathematical preliminaries that underpin the fractional derivatives and transforms used in the subsequent numerical algorithms.

5.1 Basic definitions

Fractional calculus generalizes classical differentiation and integration to non-integer orders, enabling the modeling of systems with memory and hereditary properties. In this study, we focus on several fundamental operators:

Generalized Fractional Integral: For continuous functions f, the left-side generalized fractional integral (FI), denoted by $\mathbf{I}_{a+}^{\alpha,\rho}f(t), \alpha > 0$, and $\rho > 0$, is given by

$$\mathbf{I}_{a+}^{\alpha,\rho}f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}f(s)ds, \alpha > 0, t > a \dots. (1)$$

for $m - 1 < \alpha \leq m$ where $m \in \mathbb{N}$.

Generalized Riemann-Liouville Fractional Derivative (RLFD): For continuous functions f, the generalized fractional derivative (RLFD) denoted by ${}^{R}\mathbf{D}_{a+}^{\alpha,\rho}f(t)$, of order $\alpha > 0$ is given by

$${}^{R}\mathbf{D}_{a+}^{\alpha,\rho}f(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^m \int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1}f(s)ds, t > a \geq 0 \dots (2)$$

Generalized Caputo-Type Fractional Derivative (CFD): For continuous functions f, the generalized fractional derivative of the Caputo type (CFD), denoted by ${}^{C}\mathbf{D}_{a+}^{\alpha,\rho}f(t)$, of order $\alpha > 0$ is given by

$${}^{C}\mathbf{D}_{a+}^{\alpha,\rho}f(t) = {}^{R}\mathbf{D}_{a+}^{\alpha,\rho}\left(f(x) - \sum_{n=0}^{m-1}\frac{f^{(n)}(a)}{n!}(x-a)^n\right)(t), t > a \geq 0, \dots. (3)$$

where $m = \lceil\alpha\rceil$ and $\rho > 0$. In case of $0 < \alpha \leq 1$.

$${}^{C}\mathbf{D}_{a+}^{\alpha,\rho}f(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_a^t (t^\rho - s^\rho)^{-\alpha}s^{1-\rho}f'(s)ds, 0 < \alpha \leq 1, t > a \geq 0 \dots (4)$$

The new generalized (CFD) operator, $\mathbf{D}_{a+}^{\alpha,\rho}, \alpha > 0$ is given by:

$$\mathbf{D}_{a+}^{\alpha,\rho}f(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{m-\alpha-1}\left(s^{1-\rho}\frac{d}{ds}\right)^m f(s)ds, t > a, \dots. (5)$$

where $\rho > 0, a \geq 0$, and $m - 1 < \alpha < m$.

Generalized Laplace Transform: if $f: [0,\infty] \to \mathbb{R}$, then the generalized Laplace transform of f is defined by

$$\mathcal{T}_\rho\{f(t)\} = \int_0^\infty e^{-\delta t^\rho}f(t)t^{\rho-1}dt \dots (6)$$

The generalized Laplace transform of the generalized (CFD) $\mathbf{D}_0^{\alpha,\rho}$ is defined by

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$$\mathcal{T}_\rho\{\mathbf{D}_0^{\alpha,\rho}f(t)\} = \delta^\alpha\mathcal{T}_\rho\{f(t)\} - \delta^{-1}f(0), 0 < \alpha \le 1 \dots\dots (7)$$

These definitions provide the mathematical foundation for the advanced numerical algorithms developed and investigated in this research. The following subsections detail the implementation of the adaptive predictor-corrector and Laplace decomposition methods, followed by their application to representative fractional-order systems and a critical discussion of the results.

5.2 Algorithm of the adaptive predictor-corrector method

The adaptive predictor-corrector (P-C) method is a powerful algorithm developed for the efficient numerical solution of initial value problems involving generalized Caputo-type fractional derivatives. This technique is particularly suitable for fractional-order differential equations, where classical numerical methods often struggle with convergence and stability due to memory and hereditary effects. By employing a non-uniform grid and leveraging the specific properties of the generalized Caputo derivative, the adaptive P-C method achieves high accuracy and computational efficiency. The following section outlines the mathematical foundation of the method, details the stepwise algorithm, and explains its implementation for solving fractional initial value problems.

$$\mathbf{D}_{a+}^{\alpha,\rho}y(t) = f(t, y(t)), t \in [0, T], y^{(k)}(a) = y_0^k, k = 0,1,\cdots, \lceil\alpha\rceil, \dots (8)$$

where $\mathbf{D}_{a+}^{\alpha,\rho}$ is CFD, for $m - 1 < \alpha \le m, a \ge 0, \rho > 0$ and $y \in C^m([a, T])$, the IVP (8) is equivalent, we get:

$$y(t) = u(t) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}f(s, y(s))ds \dots (9)$$

where

$$u(t) = \sum_{n=0}^{m-1}\frac{1}{\rho^n n!}(t^\rho - a^\rho)^n\left(x^{1-\rho}\frac{d}{dx}\right)^n y(x)\bigg|_{x=a} \dots. (10)$$

we partition the interval into N unequal subintervals $[t_k, t_{k+1}], k = 0,1,\cdots N-1$ using mesh points.

$$t_0 = a, t_{k+1} = \left(t_k^\rho + h\right)^{1/\rho}, k = 0,1,\cdots, N-1, \dots (11)$$

where $h = \frac{T^\rho - a^\rho}{N}$. [54] $y_k, k = 0,1,\cdots, N$, then $y(t_j)$ and $y_j \approx y(t_j)(j = 1,2,\cdots, k)$, using the integral equation. $y_{k+1} \approx y(t_{k+1})$.

$$y(t_{k+1}) = u(t_{k+1}) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{t_{k+1}} s^{\rho-1}\left(t_{k+1}^\rho - s^\rho\right)^{\alpha-1}f(s, y(s))ds.. (12)$$

Now, exchange

$$z = s^\rho, \dots (13)$$

we attain

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$$y(t_{k+1}) = u(t_{k+1}) + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_{a^\rho}^{t_{k+1}^\rho} \left(t_{k+1}^\rho - z\right)^{\alpha-1} f\left(z^{1/\rho}, y\left(z^{1/\rho}\right)\right) dz \ldots (14)$$

That is

$$y(t_{k+1}) = u(t_{k+1}) + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j^\rho}^{t_{j+1}^\rho} \left(t_{k+1}^\rho - z\right)^{\alpha-1} f\left(z^{1/\rho}, y\left(z^{1/\rho}\right)\right) dz \ldots . (15)$$

Following that, an submission of trapezoidal quadrature method by respect to the weight meaning $\left(t_{k+1}^\rho - z\right)^{\alpha-1}$, we get hold of the corrector formula to get a close rough calculation of the right-hand side of Eq. (15),

$$y(t_{k+1}) \approx u(t_{k+1}) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f\left(t_j, y(t_j)\right) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} f\left(t_{k+1}, y(t_{k+1})\right) \ldots . (16)$$

where

$$a_{j,k+1} = \begin{cases} \left(k^{\alpha+1} - (k-\alpha)(k+1)^\alpha\right) & \text{if } j = 0 \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1} & \text{if } 1 \le j < k \end{cases} \ldots (17)$$

Equation (14) is evaluate by deputy $y(t_{k+1})$ with the analyst $y^P(t_{k+1})$ by Eq. (16) where $f\left(z^{1/\rho}, y\left(z^{1/\rho}\right)\right)$ is put back by $f\left(t_j, y(t_j)\right)$.

$$y^P(t_{k+1}) \approx u(t_{k+1}) + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j^\rho}^{t_{j+1}^\rho} \left(t_{k+1}^\rho - z\right)^{\alpha-1} f\left(t_j, y(t_j)\right) dz \qquad (18)$$

$$= u(t_{k+1}) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{k} \left((k+1-j)^\alpha - (k-j)^\alpha\right) f\left(t_j, y(t_j)\right) \qquad (18)$$

after that,

$$y_{k+1} \approx u(t_{k+1}) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + \frac{\rho^{-\alpha} h^\alpha}{\Gamma(\alpha+2)} f\left(t_{k+1}, y_{k+1}^p\right). \qquad (19)$$

The proposed adaptive (P-C) method use a non-uniform grid $t_{j+1} = \left(t_j^\rho + h_\rho\right)^{1/\rho}: j = 0,1,\cdots,N-1, t_0 = a$ and $h = \frac{T^\rho - a^\rho}{N}$, where N is natural. In summary, the adaptive predictor-corrector method offers a systematic and robust framework for solving fractional-order differential equations with generalized Caputo-type derivatives. Its adaptive grid structure, integration of predictor and corrector steps, and compatibility with memory-dependent dynamics make it highly effective for both theoretical studies and practical applications. The method's flexibility and accuracy are critical in modeling complex systems, ensuring reliable solutions even in cases where conventional algorithms may fail. As demonstrated in subsequent sections, this approach provides a foundational tool for advancing the numerical analysis of fractional-order systems across various scientific and engineering domains.

6. Conclusion

This study has developed and rigorously validated two advanced numerical methods the adaptive predictor-corrector (P-C) method and the generalized Laplace decomposition method (TρDM) for efficiently solving fractional-order differential systems involving generalized Caputo-type derivatives. By applying these approaches to both the Abel differential equation and the four-dimensional Chen system, we have demonstrated their high accuracy, computational efficiency, and flexibility in capturing a broad spectrum of dynamical behaviors, including chaos. Comparative analysis with the Adams-Bashforth-Moulton (ABM) method confirms that the proposed algorithms consistently deliver results of comparable or superior quality, while offering practical advantages in terms of adaptability and ease of implementation. Collectively, these findings underscore the significance of the proposed numerical techniques as reliable tools for modeling, analysis, and simulation of complex fractional-order systems in science and engineering.

7. Future Scope

The promising results of this work suggest several directions for future research. First, the adaptive P-C and TρDM methods could be extended to handle higher-dimensional and multi-physics fractional systems, accommodating variable-order and distributed-order derivatives. Second, the integration of these schemes with machine learning and data-driven modeling techniques could further enhance their predictive capability and applicability to data-rich, real-world problems. Third, there is scope for exploring their utility in domains such as biomedical signal processing, advanced control systems, and networked dynamical systems. Additionally, the development of parallelized and GPU-accelerated algorithms could address computational bottlenecks in large-scale simulations. Finally, further theoretical work on convergence, stability, and error analysis can provide deeper mathematical foundation for these and related methods.

8. Limitations

Despite its contributions, the present study has certain limitations. The numerical experiments were restricted to specific benchmark systems with fixed fractional orders, and the methods' performance for stiff, high-dimensional, or variable-order systems has not been exhaustively explored. The analysis primarily focused on deterministic systems, whiles the extension to stochastic or spatially distributed fractional models remains an open challenge. Computational efficiency, though improved compared to classical methods, may still be a concern for very large-scale or real-time applications. Additionally, the practical implementation may require

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careful tuning of parameters such as mesh size or order, which could affect convergence and accuracy in challenging scenarios. Addressing these limitations will be crucial for the broader adoption and robustness of advanced numerical methods in fractional calculus.

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